

COPIES OF THE RANDOM GRAPH: THE 2-LOCALIZATION

Miloš S. Kurilić¹ and Stevo Todorčević²

Abstract

Let G be a countable graph containing a copy of the countable random graph (Erdős-Rényi graph, Rado graph), $\text{Emb}(G)$ the monoid of its self-embeddings, $\mathbb{P}(G) = \{f[G] : f \in \text{Emb}(G)\}$ the set of copies of G contained in G , and \mathcal{I}_G the ideal of subsets of G which do not contain a copy of G . We show that the poset $\langle \mathbb{P}(G), \subset \rangle$, the algebra $P(G)/\mathcal{I}_G$, and the inverse of the right Green's pre-order $\langle \text{Emb}(G), \preceq^R \rangle$ have the 2-localization property. The Boolean completions of these pre-orders are isomorphic and satisfy the following law: for each double sequence $[b_{nm} : \langle n, m \rangle \in \omega \times \omega]$ of elements of \mathbb{B}

$$\bigwedge_{n \in \omega} \bigvee_{m \in \omega} b_{nm} = \bigvee_{\mathcal{T} \in \text{Bt}(<^\omega \omega)} \bigwedge_{n \in \omega} \bigvee_{\varphi \in \mathcal{T} \cap^{n+1} \omega} \bigwedge_{k \leq n} b_{k\varphi(k)},$$

where $\text{Bt}(<^\omega \omega)$ denotes the set of all binary subtrees of the tree $<^\omega \omega$.

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1 Introduction

The structure considered in this paper is the self-embedding monoid $\text{Emb}(R)$ of the countable random graph (the Erdős-Rényi graph [2], the Rado graph [14]).

More generally, as a part of the investigation of the class of self-embedding monoids of first order structures, in an attempt to classify the pre-orders of the form $\langle \text{Emb}(\mathbb{X}), \preceq^R \rangle$, where \mathbb{X} is a structure and \preceq^R the right Green's pre-order on the set $\text{Emb}(\mathbb{X})$ of its self-embeddings (defined by $f \preceq^R g$ iff $f \circ h = g$, for some $h \in \text{Emb}(\mathbb{X})$), one can define two such pre-orders to be equivalent iff the antisymmetric quotients of their inverses have isomorphic Boolean completions. It is easy to see [10] that the antisymmetric quotient of the pre-order $\langle \text{Emb}(\mathbb{X}), (\preceq^R)^{-1} \rangle$ is isomorphic to the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \text{Emb}(\mathbb{X})\}$ is the set of copies of \mathbb{X} (that is, the set of domains of the substructures of \mathbb{X} which are isomorphic to \mathbb{X}) and that the isomorphism of the Boolean completions of such

¹Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia. email: milos@dmf.uns.ac.rs

²Institut de Mathématique de Jussieu (UMR 7586) Case 247, 4 Place Jussieu, 75252 Paris Cedex, France and Department of Mathematics, University of Toronto, Toronto, Canada M5S 2E4. email: stevo.todorcevic@imf-prg.fr and stevo@math.toronto.edu

posets is the same as their forcing equivalence [9]. Thus, the intended classification is in fact the classification of the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ determined by their forcing-related properties. Clearly, this classification induces a coarse classification of structures as well (see [4, 5, 6, 7, 8], for countable relational structures).

Concerning the countable ultrahomogeneous relational structures first we mention the following result from [11] related to the poset of copies of the rational line, \mathbb{Q} , and the corresponding quotient $P(\mathbb{Q})/\text{Scatt}$, where Scatt denotes the ideal of scattered suborders of \mathbb{Q} : if \mathbb{S} denotes the Sacks perfect set forcing and $\text{sh}(\mathbb{S})$ the size of the continuum in the Sacks extension, then for each countable non-scattered linear order L and, in particular, for the rational line, the poset $\mathbb{P}(L)$ is forcing equivalent to the two-step iteration

$$\mathbb{S} * \pi$$

where $1_{\mathbb{S}} \Vdash \text{“}\pi \text{ is a } \sigma\text{-closed forcing”}$. If the equality $\text{sh}(\mathbb{S}) = \aleph_1$ (implied by CH) or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega)/\text{Fin})^+$ of the Sacks extension. Consequently,

$$\text{ro sq asq} \langle \text{Emb}(\mathbb{Q}), (\preceq^R)^{-1} \rangle \cong \text{ro sq } \mathbb{P}(\mathbb{Q}) \cong \text{ro}((P(\mathbb{Q})/\text{Scatt})^+) \cong \text{ro}(\mathbb{S} * \pi).$$

The following similar statement for countable non-scattered graphs (that is, the graphs containing a copy of the Rado graph) was obtained in [12].

Theorem 1.1 *For each countable non-scattered graph $\langle G, \sim \rangle$ and, in particular, for the Rado graph, the poset $\mathbb{P}(G)$ is forcing equivalent to the two-step iteration*

$$\mathbb{P} * \pi \tag{1}$$

where $1_{\mathbb{P}} \Vdash \text{“}\pi \text{ is an } \omega\text{-distributive forcing”}$ and the poset \mathbb{P} adds a generic real, has the \aleph_0 -covering property (thus preserves ω_1), has the Sacks property and does not produce splitting reals. In addition,

$$\text{ro sq asq} \langle \text{Emb}(G), (\preceq^R)^{-1} \rangle \cong \text{ro sq } \mathbb{P}(G) \cong \text{ro}(P(R)/\mathcal{I}_R)^+ \cong \text{ro}(\mathbb{P} * \pi) \tag{2}$$

and these complete Boolean algebras are weakly distributive³.

³A complete Boolean algebra \mathbb{B} is called *weakly distributive* (or $(\omega, \cdot, < \omega)$ -distributive, see [3]) iff for each cardinal κ and each matrix $[b_{n\alpha} : \langle n, \alpha \rangle \in \omega \times \kappa]$ of elements of \mathbb{B} we have

$$\bigwedge_{n \in \omega} \bigvee_{\alpha \in \kappa} b_{n\alpha} = \bigvee_{s: \omega \rightarrow [\kappa] < \omega} \bigwedge_{n \in \omega} \bigvee_{\alpha \in s(n)} b_{n\alpha}.$$

We note that the Sacks forcing has all the properties listed in Theorem 1.1: adds a generic real, has the \aleph_0 -covering and the Sacks property and does not produce splitting reals. In the present paper we show that the poset of copies of the Rado graph and, hence, the forcing \mathbb{P} from (1) shares one more property with the Sacks forcing - the 2-localization property. In order to define it we recall that a subtree \mathcal{T} of the tree ${}^{<\omega}\omega = \bigcup_{n \in \omega} {}^n\omega$ is called *binary* iff each $\varphi \in \mathcal{T}$ has at most two immediate successors in \mathcal{T} ; by $\text{Bt}({}^{<\omega}\omega)$ we will denote the set of all binary subtrees of ${}^{<\omega}\omega$. A pre-order \mathbb{P} is said to have the *2-localization property* iff in each generic extension of the ground model V by \mathbb{P} we have: for each function $x : \omega \rightarrow \omega$ there is a binary subtree \mathcal{T} of ${}^{<\omega}\omega$ belonging to V and such that the set of finite approximations of x , $\{x \upharpoonright n : n \in \omega\}$, is a branch in \mathcal{T} , that is

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} \forall x : \check{\omega} \rightarrow \check{\omega} \ \exists \mathcal{T} \in ((\text{Bt}({}^{<\omega}\omega))^V)^{\sim} \ \forall n \in \check{\omega} \ x \upharpoonright n \in \mathcal{T}. \quad (3)$$

It is clear that the 2-localization property implies the Sacks property, but the converse is not true (see [13]). For a complete Boolean algebra \mathbb{B} , the 2-localization property has the following forcing-free translation, similar to the distributivity laws: for each double sequence $[b_{nm} : \langle n, m \rangle \in \omega \times \omega]$ of elements of \mathbb{B}

$$\bigwedge_{n \in \omega} \bigvee_{m \in \omega} b_{nm} = \bigvee_{\mathcal{T} \in \text{Bt}({}^{<\omega}\omega)} \bigwedge_{n \in \omega} \bigvee_{\varphi \in \mathcal{T} \cap {}^{n+1}\omega} \bigwedge_{k \leq n} b_{k\varphi(k)} \quad (4)$$

(see Appendix). Thus, the following statement is the main result of the paper.

Theorem 1.2 *For each countable non-scattered graph $\langle G, \sim \rangle$ and, in particular, for the Rado graph, the poset $\mathbb{P}(G)$, and, hence, the first iterand \mathbb{P} from (1), has the 2-localization property. Consequently, the algebras from (2) satisfy (4).*

2 Preliminaries

First we introduce a convenient notation. If $\langle G, \sim \rangle$ is a graph (namely, if \sim is a symmetric and irreflexive binary relation on the set G) and $K \subset H \in [G]^{<\omega}$, let

$$G_K^H := \left\{ v \in G \setminus H : \forall k \in K (v \sim k) \wedge \forall h \in H \setminus K (v \not\sim h) \right\}.$$

(Clearly, $G_{\emptyset}^{\emptyset} = G$.) The object of our study is the Rado graph, characterized as the unique (up to isomorphism) countable graph $\langle R, \sim \rangle$ such that

$$R_K^H \neq \emptyset, \text{ whenever } K \subset H \in [R]^{<\omega}. \quad (5)$$

More information about the Rado graph and the related structures can be found in the survey article [1]. Now we recall some definitions and facts which will be used in the sequel.

Fact 2.1 Let $\langle R, \sim \rangle$ be the Rado graph and $\mathbb{P}(R)$ the set of its copies. Then

- (a) If F is a finite subset of R , then $R \setminus F \in \mathbb{P}(R)$;
- (b) If $\{X_1, \dots, X_k\}$ is a partition of R , then $X_i \in \mathbb{P}(R)$, for some $i \leq k$ (namely, the Rado graph is a strongly indivisible structure);
- (c) If H is a finite subset of R , then $\{H\} \cup \{R_K^H : K \subset H\}$ is a partition of R and $R_K^H \in \mathbb{P}(R)$, for each $K \subset H$.

Fact 2.2 ([12]) Let H_1 and H_2 be finite subsets of R , $K_1 \subset H_1$ and $K_2 \subset H_2$.

- (a) $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} \neq \emptyset$ if and only if $H_1 \cap K_2 = H_2 \cap K_1$;
- (b) $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} \neq \emptyset$ implies that $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} = R_{K_1 \cup K_2}^{H_1 \cup H_2}$;
- (c) $R_{K_1}^{H_1} = R_{K_2}^{H_2}$ if and only if $H_1 = H_2$ and $K_1 = K_2$;
- (d) $R_{K_1}^{H_1} \subset R_{K_2}^{H_2}$ if and only if $H_1 \supset H_2$, $K_1 \supset K_2$ and $H_2 \cap K_1 = K_2$.

If $\langle R, \sim \rangle$ is the Rado graph and $L \in \mathbb{P}(R)$, we will say that a pair $\mathcal{L} = \langle \Pi, q \rangle$ is a *labeling* of L iff

- (L1) $\Pi = \{L_n : n \in \omega\}$ is a partition of the set L ,
- (L2) $q : \bigcup_{n \in \omega} \{n\} \times P(\bigcup_{i < n} L_i) \rightarrow L$ is a bijection,
- (L3) $L_n = \{q(n, K) : K \subset \bigcup_{i < n} L_i\}$, for each $n \in \omega$,
- (L4) $q(n, K) \in L_K^{\bigcup_{i < n} L_i}$, for each $n \in \omega$ and each $K \subset \bigcup_{i < n} L_i$.

Then, clearly, $L_0 = \{q(0, \emptyset)\}$, $|L_0| = 1$ and the sets L_n are finite. More precisely, by (L3) we have $|L_n| = m_n$, where the integers m_n , $n \in \omega$, are defined by: $m_0 = 1$ and $m_n = 2^{\sum_{i < n} m_i}$, for $n > 0$. Thus $\langle |L_n| : n \in \omega \rangle = \langle 1, 2, 8, 2^{11}, \dots \rangle$. We note that, by [12], each copy $L \in \mathbb{P}(R)$ has infinitely many labelings. For convenience, instead of $q(n, K)$ we will write $q_K^{\bigcup_{i < n} L_i}$ and the labeling \mathcal{L} will be denoted by

$$\left\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \right\rangle.$$

If $\mathcal{D} = \langle \mathcal{D}_n : n \in \omega \rangle$ is a sequence of subsets of $\mathbb{P}(R)$, then a copy $L \in \mathbb{P}(R)$ will be called a *fusion* of the sequence \mathcal{D} if and only if there exists a labeling $\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ of L such that

$$\forall n \in \omega \quad \forall K \subset \bigcup_{i < n} L_i \quad \exists D \in \mathcal{D}_n \quad L_K^{\bigcup_{i < n} L_i} \subset D. \quad (6)$$

Fact 2.3 ([12]) *If $\mathcal{D} = \langle \mathcal{D}_n : n \in \omega \rangle$ is a sequence subsets of $\mathbb{P}(R)$ which are dense below $A \in \mathbb{P}(R)$, then the set $\mathcal{F} = \{L : L \text{ is a fusion of } \mathcal{D}\}$ is dense below A .*

Using a labeling $\mathcal{L} = \langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ of a copy $L \in \mathbb{P}(R)$ we define a partial ordering $\leq_{L, \mathcal{L}}$ on L by:

$$q_{K'}^{\bigcup_{i < m} L_i} \leq_{L, \mathcal{L}} q_{K''}^{\bigcup_{i < n} L_i} \Leftrightarrow L_{K'}^{\bigcup_{i < m} L_i} \subset L_{K''}^{\bigcup_{i < n} L_i} \quad (7)$$

$$\Leftrightarrow m \geq n \wedge K' \cap \bigcup_{i < n} L_i = K'' \quad (8)$$

(see Fact 2.2(c) and (d)). Writing shortly \leq_L instead of $\leq_{L, \mathcal{L}}$ we have

Fact 2.4 ([12]) *$\langle L, \leq_L \rangle$ is a finitely branching reversed tree without minimal nodes, with the top q_\emptyset^0 , and the set L_n is its n -th level. For each $n \in \omega$ and $K \subset \bigcup_{i < n} L_i$ we have*

$$\left(-\infty, q_K^{\bigcup_{i < n} L_i} \right]_{\langle L, \leq_L \rangle} = L_K^{\bigcup_{i < n} L_i}.$$

3 The 2-localization property

In this section we prove Theorem 1.2. In fact, if $\langle G, \sim \rangle$ is a countable graph containing a copy of the Rado graph, then these two structures are equimorphic and, by [9], the corresponding posets of copies $\mathbb{P}(G)$ and $\mathbb{P}(R)$ are forcing equivalent. So it is sufficient to prove the theorem assuming that $\langle G, \sim \rangle$ is the Rado graph and we will do this in Theorem 3.3. We start with two auxiliary statements.

Lemma 3.1 *Let $k_n \in \omega$, $n \in \omega$, where $k_0 = 0$ and $k_{n+1} = k_n + 2^{n+1} - 1$, and let ${}^{n+1}2 = \{\varphi_i^n : i < 2^n\}$ be an enumeration of the n -th level of the binary tree ${}^{<\omega}2 = \bigcup_{n \in \omega} {}^n2$ induced by the lexicographic order (for example: $\varphi_0^2 = 00$, $\varphi_1^2 = 01$, $\varphi_2^2 = 10$, and $\varphi_3^2 = 11$). Then*

$$\Delta = \left\{ \varphi_i^{n+1} \frown \underbrace{00 \dots 0}_{k_n + i} : n \in \omega \wedge i < 2^{n+1} \right\}$$

is a dense subset of the reversed tree $\langle {}^{<\omega}2, \supset \rangle$ and $|\Delta \cap {}^n2| = 1$, for each $n \in \mathbb{N}$.

Proof. The density of Δ is evident. Since the lengths of the elements of Δ below the $(n+1)$ -th level of ${}^{<\omega}2$ run from $n+1+k_n$ to $n+1+k_n+2^{n+1}-1 = n+1+k_{n+1}$, while the lengths of the elements of Δ below the $(n+2)$ -th level of ${}^{<\omega}2$ start from $n+2+k_{n+1}$, the second statement is true as well. \square

Lemma 3.2 *If $g : \langle {}^{<\omega}2, \supset \rangle \rightarrow \langle {}^{<\omega}\omega, \supset \rangle$ is an embedding (that is, an injective strong homomorphism), then the set $g[{}^{<\omega}2] \upharpoonright = \{\psi \in {}^{<\omega}\omega : \exists \varphi \in {}^{<\omega}2 \ g(\varphi) \supset \psi\}$ is a binary reversed subtree of the reversed tree $\langle {}^{<\omega}\omega, \supset \rangle$.*

Proof. Suppose that $\psi \in g[{}^{<\omega}2] \uparrow$ has 3 immediate predecessors $\psi_i, i < 3$, in $g[{}^{<\omega}2] \uparrow$. Let $\varphi_i, i < 3$, be the elements of ${}^{<\omega}2$ of minimal length satisfying $g(\varphi_i) \supset \psi_i$. Then $g(\varphi_0 \cap \varphi_1) \subset \psi_0 \cap \psi_1 = \psi$. Let φ be the element of ${}^{<\omega}2$ of maximal length satisfying $\psi \supset g(\varphi)$. Suppose that for some $i < 3$ there is $\eta \in {}^{<\omega}2$ such that $\varphi_i \not\supseteq \eta \not\supseteq \varphi$. Then $g(\varphi_i) \not\supseteq g(\eta) \not\supseteq g(\varphi)$ and, since $g(\varphi_i) \not\supseteq g(\eta)$, ψ and $\langle {}^{<\omega}\omega, \supset \rangle$ is a reversed tree, $g(\eta)$ and ψ are comparable. But $\psi \supset g(\eta)$ is not true, by the maximality of φ and $g(\eta) \not\supseteq \psi$ is not true, by the minimality of φ_i . Thus $\varphi_i, i < 3$, are immediate predecessors of φ in ${}^{<\omega}2$, which is impossible. \square

Theorem 3.3 *The partial ordering $\mathbb{P}(R)$ has the 2-localization property.*

Proof. Let G be a $\mathbb{P}(R)$ -generic filter over the ground model V and $x \in V_{\mathbb{P}(R)}[G]$, where $x : \omega \rightarrow \omega$. Let τ be a $\mathbb{P}(R)$ -name such that $x = \tau_G$ and let $A \in G$, where $A \Vdash \tau : \check{\omega} \rightarrow \check{\omega}$. First we define the sets $\mathcal{D}_0, \mathcal{D}_1 \subset \mathbb{P}(R)$ by

$$\mathcal{D}_0 = \{B \in \mathbb{P}(A) : \forall n \in \omega \ \exists m \in \omega \ B \Vdash \tau(\check{n}) = \check{m}\} \text{ and}$$

$$\mathcal{D}_1 = \{C \in \mathbb{P}(A) : \forall B \in \mathbb{P}(C) \ \exists n \in \omega \ \forall m \in \omega \ \neg B \Vdash \tau(\check{n}) = \check{m}\}$$

and show that, in the poset $\langle \mathbb{P}(R), \subset \rangle$, the union $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$ is dense below A . So, let $D \in \mathbb{P}(R)$ and $D \subset A$. If $D \in \mathcal{D}_1$, we are done; otherwise, there is $B \in \mathbb{P}(D)$ such that $B \in \mathcal{D}_0$, thus $\mathcal{D} \ni B \subset D$ and \mathcal{D} is dense below A indeed.

Since $A \in G$, the genericity of G implies $G \cap \mathcal{D} \neq \emptyset$ and we have two cases.

Case 1: $G \cap \mathcal{D}_0 \neq \emptyset$. Then $B \in G$, for some $B \in \mathcal{D}_0$, and, by the definability of the forcing relation, the set $f = \{\langle n, m \rangle \in \omega \times \omega : B \Vdash \tau(\check{n}) = \check{m}\}$ belongs to V . In order to prove that $B \Vdash \tau = \check{f}$ we assume that H is a $\mathbb{P}(R)$ -generic filter over V containing B and show that $\tau_H = f$. If $\langle n, m \rangle \in \tau_H$, then, since $B \in \mathcal{D}_0$, there is $p \in \omega$ such that $B \Vdash \tau(\check{n}) = \check{p}$, which implies $\langle n, p \rangle \in \tau_H$; thus, since $B \Vdash \tau : \check{\omega} \rightarrow \check{\omega}$, we have $p = m$ and, hence $\langle n, m \rangle \in f$. Conversely, if $\langle n, m \rangle \in f$, then $B \Vdash \tau(\check{n}) = \check{m}$ and, since $B \in H$, we have $\langle n, m \rangle \in \tau_H$. So $B \Vdash \tau = \check{f}$ and, by the assumption, $B \in G$, which implies $\tau_G = f \in V$. Thus for $\mathcal{T} = \{f \upharpoonright n : n \in \omega\}$ we have $\mathcal{T} \in \text{Bt}({}^{<\omega}\omega)$ and $\tau_G \upharpoonright n \in \mathcal{T}$, for all $n \in \omega$.

Case 2: $G \cap \mathcal{D}_0 = \emptyset$. Then, by the density of \mathcal{D} , there is $C \in G \cap \mathcal{D}_1$ so we have

$$\forall B \in \mathbb{P}(C) \ \exists n \in \omega \ \forall m \in \omega \ \neg B \Vdash \tau(\check{n}) = \check{m} \quad (9)$$

and, since $C \in G$, it is sufficient to prove that

$$\forall B \in \mathbb{P}(C) \ \exists D \in \mathbb{P}(B) \ \exists \mathcal{T} \in \text{Bt}({}^{<\omega}\omega) \ \forall n \in \omega \ D \Vdash \tau \upharpoonright n \in \check{\mathcal{T}}. \quad (10)$$

Let $B \in \mathbb{P}(C)$. Then, since $B \subset C \subset A$, we have $B \Vdash \tau : \check{\omega} \rightarrow \check{\omega}$ and, hence, $B \Vdash \forall n \in \check{\omega} \ \exists m \in \check{\omega} \ \tau(n) = m$. Thus, for each $n \in \omega$ we have: for each

$B' \in \mathbb{P}(B)$ there are $D \in \mathbb{P}(B')$ and $m \in \omega$ such that $D \Vdash \tau(\check{n}) = \check{m}$. This means that the sets $\mathcal{D}_n := \{D \in \mathbb{P}(B) : \exists m \in \omega \ D \Vdash \tau(\check{n}) = \check{m}\}$, $n \in \omega$, are dense below B . By Fact 2.3, the set \mathcal{F} of fusions of the sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ is dense below B and, hence, there is a fusion $L \in \mathcal{F}$ such that $L \in \mathbb{P}(B)$. So there is a labeling $\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ of L such that, by (6), for $n \in \omega$ and $K \subset \bigcup_{i < n} L_i$ there is $D \in \mathcal{D}_n$ such that $L_K^{\bigcup_{i < n} L_i} \subset D$ and, hence, there is (clearly unique) $m \in \omega$ satisfying $L_K^{\bigcup_{i < n} L_i} \Vdash \tau(\check{n}) = \check{m}$. Thus we obtain an indexed family of integers $\{m_K^n : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\}$ such that

$$\forall n \in \omega \ \forall K \subset \bigcup_{i < n} L_i \ L_K^{\bigcup_{i < n} L_i} \Vdash \tau(\check{n}) = \check{m}_K^n. \quad (11)$$

Let \leq_L be the reversed tree order on L defined by (7) or, equivalently, by (8).

Claim 1 *For each $n \in \omega$ and each $K \subset \bigcup_{i < n} L_i$ there are $n_1 > n$ and $K', K'' \subset \bigcup_{i < n_1} L_i$ such that*

$$q_{K'}^{\bigcup_{i < n_1} L_i}, q_{K''}^{\bigcup_{i < n_1} L_i} <_L q_K^{\bigcup_{i < n} L_i} \ \wedge \ m_{K'}^{n_1} \neq m_{K''}^{n_1}. \quad (12)$$

Proof. Suppose, contrary to our claim, that there are $n_0 \in \omega$ and $K_0 \subset \bigcup_{i < n_0} L_i$ such that for each $n > n_0$ and each $K', K'' \subset \bigcup_{i < n} L_i$ we have (see (8))

$$K' \cap \bigcup_{i < n_0} L_i = K'' \cap \bigcup_{i < n_0} L_i = K_0 \Rightarrow m_{K'}^n = m_{K''}^n. \quad (13)$$

Then for each $n > n_0$ there is $m(n) \in \omega$ such that for each $K \subset \bigcup_{n_0 \leq i < n} L_i$ we have $m_{K_0 \cup K}^n = m(n)$ which, by (11), implies $L_{K_0 \cup K}^{\bigcup_{i < n} L_i} = (L_{K_0}^{\bigcup_{i < n_0} L_i})_K^{\bigcup_{n_0 \leq i < n} L_i} \Vdash \tau(\check{n}) = \check{m}(n)$. By Fact 2.1(c) $\{(L_{K_0}^{\bigcup_{i < n_0} L_i})_K^{\bigcup_{n_0 \leq i < n} L_i} : K \subset \bigcup_{n_0 \leq i < n} L_i\}$ is an antichain in $\mathbb{P}(R)$, maximal below $L_{K_0}^{\bigcup_{i < n_0} L_i}$, thus $L_{K_0}^{\bigcup_{i < n_0} L_i} \Vdash \tau(\check{n}) = \check{m}(n)$ and this holds for all $n > n_0$. For $n \leq n_0$, let $m(n) = m_{K_0 \cap \bigcup_{i < n} L_i}^{\bigcup_{i < n} L_i}$; then, since $L_{K_0}^{\bigcup_{i < n_0} L_i} \subset L_{K_0 \cap \bigcup_{i < n} L_i}^{\bigcup_{i < n} L_i}$, by (11) we have $L_{K_0}^{\bigcup_{i < n_0} L_i} \Vdash \tau(\check{n}) = \check{m}(n)$. Thus $L_{K_0}^{\bigcup_{i < n_0} L_i} \in \mathbb{P}(C)$ and $L_{K_0}^{\bigcup_{i < n_0} L_i} \Vdash \tau(\check{n}) = \check{m}(n)$, for each $n \in \omega$, which is not true by (9). \square

Strategy. Roughly speaking, in order to prove (10), using the fusion $L \subset B$ obtained above, its labeling and the corresponding order \leq_L , we will construct:

- a copy Λ of the reversed binary tree $\langle {}^{<\omega}2, \supset \rangle$ in the reversed tree $\langle L, \leq_L \rangle$,
- a copy D of the Rado graph contained in Λ ,
- an embedding F of Λ in the reversed tree $\langle {}^{<\omega}\omega, \supset \rangle$ such that for the upwards closure \mathcal{T} of $F[\Lambda]$ we have $D \Vdash \{\tau \restriction n : n \in \check{\omega}\} \subset \check{\mathcal{T}}$.

Construction. Let $\Delta = \{\psi_n : n \in \mathbb{N}\}$ be the enumeration of the set $\Delta \subset {}^{<\omega}2$, defined in Lemma 3.1, such that $\Delta \cap {}^n2 = \{\psi_n\}$. For convenience let $\Lambda_0 := \{q_\emptyset^\emptyset\}$, $l_0 := 0$ and $K_\emptyset := \emptyset$. Using recursion, for each $n \in \mathbb{N}$ we define $\Lambda_n, f_n, l_n, d_n, \Phi_n, \{n_\varphi : \varphi \in {}^{n-1}2\}$ and $\{K_\varphi : \varphi \in {}^n2\}$ such that for each $n \in \mathbb{N}$ we have:

- (i) $K_{\varphi \frown j} \subset \bigcup_{i < n_\varphi} L_i$, for each $\varphi \in {}^{n-1}2$ and $j \in 2$;
- (ii) $n_\varphi \leq n_{\psi_n \upharpoonright (n-1)} < n_{\psi_n \upharpoonright (n-1)} + 1 = l_n < n_{\varphi \frown j}$, for $\varphi \in {}^{n-1}2$ and $j \in 2$;
- (iii) $q_{K_{\varphi \frown 0 \frown j}}^{\bigcup_{i < n_{\varphi \frown 0}} L_i} <_L q_{K_{\varphi \frown 0}}^{\bigcup_{i < l_n} L_i}$, for each $\varphi \in {}^{n-1}2$ and $j \in 2$;
- (iv) $q_{K_{\varphi \frown 1 \frown j}}^{\bigcup_{i < n_{\varphi \frown 1}} L_i} <_L q_{K_{\varphi \frown 1 \cup \{d_n\}}}^{\bigcup_{i < l_n} L_i}$, for each $\varphi \in {}^{n-1}2$ and $j \in 2$;
- (v) $m_{K_{\varphi \frown 0}}^{n_\varphi} \neq m_{K_{\varphi \frown 1}}^{n_\varphi}$, for each $\varphi \in {}^{n-1}2$;
- (vi) $\Lambda_n = \{q_{K_{\varphi \frown j}}^{\bigcup_{i < n_\varphi} L_i} : \varphi \in {}^{n-1}2 \wedge j \in 2\}$;
- (vii) $f_n : \langle \bigcup_{i \leq n} {}^i2, \supset \rangle \rightarrow \langle \bigcup_{i \leq n} \Lambda_i, \leq_L \rangle$ is an isomorphism, where
$$f_n(\emptyset) = q_\emptyset^\emptyset \quad \text{and} \quad f_n(\varphi \frown j) = q_{K_{\varphi \frown j}}^{\bigcup_{i < n_\varphi} L_i};$$
- (viii) $f_m \subset f_n$, for all $m \leq n$;
- (ix) $d_n = f_n(\psi_n)$;
- (x) $\Phi_n = \{q_{K_{\varphi \frown 0}}^{\bigcup_{i < l_n} L_i} : \varphi \in {}^{n-1}2\} \cup \{q_{K_{\varphi \frown 1 \cup \{d_n\}}}^{\bigcup_{i < l_n} L_i} : \varphi \in {}^{n-1}2\}$.

Claim 2 *The recursion works.*

Proof. By Claim 1 (for $n = 0$ and $K = \emptyset$) there are $n_\emptyset > 0$ and $K_0, K_1 \subset \bigcup_{i < n_\emptyset} L_i$ such that $q_{K_0}^{\bigcup_{i < n_\emptyset} L_i}, q_{K_1}^{\bigcup_{i < n_\emptyset} L_i} <_L q_\emptyset^\emptyset$ and $m_{K_0}^{n_\emptyset} \neq m_{K_1}^{n_\emptyset}$. Let $\Lambda_1 = \{q_{K_j}^{\bigcup_{i < n_\emptyset} L_i} : j \in 2\}$ let $f_1 : \langle {}^02 \cup {}^12, \supset \rangle \rightarrow \langle \Lambda_0 \cup \Lambda_1, \leq_L \rangle$, where $f_1(\emptyset) = q_\emptyset^\emptyset$ and $f_1(\langle j \rangle) = q_{K_j}^{\bigcup_{i < n_\emptyset} L_i}$, for $j \in 2$, let $l_1 = n_\emptyset + 1$, $d_1 = f_1(\psi_1) = f_1(\langle 0 \rangle) = q_{K_0}^{\bigcup_{i < n_\emptyset} L_i}$ and $\Phi_1 = \{q_{K_0}^{\bigcup_{i < l_1} L_i}, q_{K_1 \cup \{d_1\}}^{\bigcup_{i < l_1} L_i}\}$. It is easy to check that conditions (i) - (x) are satisfied.

Suppose that the objects Λ_i, f_i, l_i, d_i , and Φ_i , for $i \leq n$, $\{n_\varphi : \varphi \in {}^{\leq n-1}2\}$ and $\{K_\varphi : \varphi \in {}^{\leq n}2\}$ satisfy conditions (i) - (x).

First, for $\varphi \in {}^{n-1}2$ and $k, j \in 2$, we define $n_{\varphi \frown k}$ and $K_{\varphi \frown k \frown j}$. By (ii) and (i) we have

$$l_n > n_\varphi \quad \text{and} \quad K_{\varphi \frown k} \subset \bigcup_{i < n_\varphi} L_i \subset \bigcup_{i < l_n} L_i \quad (14)$$

Since all the sequences from Δ have 0 at the end we have $\psi_{n+1} = \varphi_0 \frown k_0 \frown 0$, for some $\varphi_0 \in {}^{n-1}2$ and some $k_0 \in 2$, and we distinguish the following four cases.

(I) $\varphi \wedge k \neq \varphi_0 \wedge k_0$ and $k = 0$. By (14) and Claim 1 (applied to l_n and $K_{\varphi \wedge 0}$) there are

$$n_{\varphi \wedge 0} > l_n \quad \text{and} \quad K_{\varphi \wedge 0 \wedge j} \subset \bigcup_{i < n_{\varphi \wedge 0}} L_i, \quad \text{for } j \in 2, \quad \text{such that} \quad (15)$$

$$q_{K_{\varphi \wedge 0 \wedge j}}^{\bigcup_{i < n_{\varphi \wedge 0}} L_i} <_L q_{K_{\varphi \wedge 0}}^{\bigcup_{i < l_n} L_i} <_L q_{K_{\varphi \wedge 0}}^{\bigcup_{i < n_{\varphi}} L_i} \quad \text{and} \quad m_{K_{\varphi \wedge 0 \wedge 0}}^{n_{\varphi \wedge 0}} \neq m_{K_{\varphi \wedge 0 \wedge 1}}^{n_{\varphi \wedge 0}}. \quad (16)$$

(II) $\varphi \wedge k \neq \varphi_0 \wedge k_0$ and $k = 1$. Then, by (ix), (vii), (vi) and (ii),

$$d_n = f_n(\psi_n) \in \Lambda_n \subset \bigcup_{\varphi' \in {}^{n-1}2} L_{n_{\varphi'}} \subset \bigcup_{i < l_n} L_i, \quad (17)$$

so, by (14) we have $K_{\varphi \wedge 1} \cup \{d_n\} \subset \bigcup_{i < l_n} L_i$ and by Claim 1 (applied to l_n and $K_{\varphi \wedge 1} \cup \{d_n\}$) there are

$$n_{\varphi \wedge 1} > l_n \quad \text{and} \quad K_{\varphi \wedge 1 \wedge j} \subset \bigcup_{i < n_{\varphi \wedge 1}} L_i, \quad \text{for } j \in 2, \quad \text{such that} \quad (18)$$

$$q_{K_{\varphi \wedge 1 \wedge j}}^{\bigcup_{i < n_{\varphi \wedge 1}} L_i} <_L q_{K_{\varphi \wedge 1} \cup \{d_n\}}^{\bigcup_{i < l_n} L_i} \quad \text{and} \quad m_{K_{\varphi \wedge 1 \wedge 0}}^{n_{\varphi \wedge 1}} \neq m_{K_{\varphi \wedge 1 \wedge 1}}^{n_{\varphi \wedge 1}}. \quad (19)$$

So $n_{\varphi \wedge k}$ and $K_{\varphi \wedge k \wedge j}$ are defined for $\varphi \wedge k \neq \varphi_0 \wedge k_0$ and now we define $n_{\varphi_0 \wedge k_0}$ and $K_{\varphi_0 \wedge k_0 \wedge j}$, for $j \in 2$. Let $n^* = \max\{n_{\varphi \wedge k} : \varphi \in {}^{n-1}2 \wedge k \in 2 \wedge \varphi \wedge k \neq \varphi_0 \wedge k_0\}$. By (15) and (18) we have $n^* > l_n$ thus, by (14),

$$K_{\varphi_0 \wedge k_0} \subset \bigcup_{i < l_n} L_i \subset \bigcup_{i < n^*} L_i. \quad (20)$$

(III) $\varphi \wedge k = \varphi_0 \wedge k_0$ and $k = 0$. By (20) and Claim 1 (applied to n^* and $K_{\varphi_0 \wedge 0}$) there are

$$n_{\varphi_0 \wedge 0} > n^* > l_n > n_{\varphi_0} \quad \text{and} \quad K_{\varphi_0 \wedge 0 \wedge j} \subset \bigcup_{i < n_{\varphi_0 \wedge 0}} L_i, \quad \text{for } j \in 2, \quad \text{where} \quad (21)$$

$$q_{K_{\varphi_0 \wedge 0 \wedge j}}^{\bigcup_{i < n_{\varphi_0 \wedge 0}} L_i} <_L q_{K_{\varphi_0 \wedge 0}}^{\bigcup_{i < l_n} L_i} <_L q_{K_{\varphi_0 \wedge 0}}^{\bigcup_{i < n_{\varphi_0}} L_i} \quad \text{and} \quad m_{K_{\varphi_0 \wedge 0 \wedge 0}}^{n_{\varphi_0 \wedge 0}} \neq m_{K_{\varphi_0 \wedge 0 \wedge 1}}^{n_{\varphi_0 \wedge 0}}. \quad (22)$$

(IV) $\varphi \wedge k = \varphi_0 \wedge k_0$ and $k = 1$. For $\varphi' \in {}^{n-1}2$ by (ii) we have $n_{\varphi'} < l_n < n^*$ and, by (17) and (20), $K_{\varphi_0 \wedge 1} \cup \{d_n\} \subset \bigcup_{i < n^*} L_i$. So, by Claim 1 (applied to n^* and $K_{\varphi_0 \wedge 1} \cup \{d_n\}$) there are

$$n_{\varphi_0 \wedge 1} > n^* > l_n \quad \text{and} \quad K_{\varphi_0 \wedge 1 \wedge j} \subset \bigcup_{i < n_{\varphi_0 \wedge 1}} L_i, \quad \text{for } j \in 2, \quad \text{such that} \quad (23)$$

$$q_{K_{\varphi_0^1 \wedge j}}^{\bigcup_{i < n} \varphi_0^1 L_i} <_L q_{K_{\varphi_0^1 \cup \{d_n\}}}^{\bigcup_{i < n^*} L_i} <_L q_{K_{\varphi_0^1 \cup \{d_n\}}}^{\bigcup_{i < l_n} L_i} \quad \text{and} \quad m_{K_{\varphi_0^1 \wedge 0}}^{n_{\varphi_0^1}} \neq m_{K_{\varphi_0^1 \wedge 1}}^{n_{\varphi_0^1}}. \quad (24)$$

So we have defined $n_{\varphi \wedge k}$ and $K_{\varphi \wedge k \wedge j}$, for $\varphi \in {}^{n-1}2$ and $k, j \in 2$. Now we define

$$\Lambda_{n+1} := \left\{ q_{K_{\varphi \wedge k \wedge j}}^{\bigcup_{i < n_{\varphi \wedge k}} L_i} : \varphi \in {}^{n-1}2 \wedge k, j \in 2 \right\}$$

and (vi) is true. (i) follows from (15), (18), (21) and (23) and (iii), (iv) and (v) follow from (16), (19), (22) and (24). Recall that $\psi_{n+1} = \varphi_0^1 k_0^1 0$ and let us define

$$l_{n+1} = n_{\varphi_0^1 k_0} + 1.$$

Then, by (21) and (23), for $\varphi \in {}^{n-1}2$ and $k \in 2$ satisfying $\varphi \wedge k \neq \varphi_0^1 k_0$ we have

$$l_{n+1} = n_{\psi_{n+1} \upharpoonright n} + 1 > n_{\psi_{n+1} \upharpoonright n} = n_{\varphi_0^1 k_0} > n^* \geq n_{\varphi \wedge k} \quad (25)$$

and, by (15), (18), (21) and (23), $n_{\varphi \wedge k} > l_n$, for all $\varphi \in {}^{n-1}2$ and $k \in 2$. Thus (ii) is true as well.

Let $f_{n+1} : \bigcup_{i \leq n+1} {}^i 2 \rightarrow \bigcup_{i \leq n+1} \Lambda_i$ be defined by: $f_{n+1} \upharpoonright \bigcup_{i \leq n} {}^i 2 = f_n$ (thus (viii) is true) and

$$f_{n+1}(\varphi \wedge k \wedge j) = q_{K_{\varphi \wedge k \wedge j}}^{\bigcup_{i < n_{\varphi \wedge k}} L_i}, \quad \text{for } \varphi \in {}^{n-1}2 \text{ and } k, j \in 2. \quad (26)$$

In order to prove that the mapping f_{n+1} is an isomorphism, first we show that

$$\forall \varphi \in {}^{n-1}2 \quad \forall k, j \in 2 \quad q_{K_{\varphi \wedge k \wedge j}}^{\bigcup_{i < n_{\varphi \wedge k}} L_i} <_L q_{K_{\varphi \wedge k}}^{\bigcup_{i < n_{\varphi}} L_i}. \quad (27)$$

In cases (I) and (III) this follows from (16) and (22). For case (II), by (19) and (8) it is sufficient to prove that $(K_{\varphi \wedge 1} \cup \{d_n\}) \cap \bigcup_{i < n_{\varphi}} L_i = K_{\varphi \wedge 1}$, and, by (i), this will follow from $d_n \notin \bigcup_{i < n_{\varphi}} L_i$. By (ix) and (vii) we have

$$d_n = f_n(\psi_n) = f_n((\psi_n \upharpoonright (n-1)) \wedge 0) = q_{K_{(\psi_n \upharpoonright (n-1)) \wedge 0}}^{\bigcup_{i < n_{\psi_n \upharpoonright (n-1)}} L_i} \in L_{n_{\psi_n \upharpoonright (n-1)}} \quad (28)$$

but, by (ii), $n_{\psi_n \upharpoonright (n-1)} \geq n_{\varphi}$ and $d_n \notin \bigcup_{i < n_{\varphi}} L_i$ indeed. For case (IV), by (24) and (8) we have to prove that $(K_{\varphi_0^1 \wedge 1} \cup \{d_n\}) \cap \bigcup_{i < n_{\varphi_0^1}} L_i = K_{\varphi_0^1 \wedge 1}$, and, by (i), this will follow from $d_n \notin \bigcup_{i < n_{\varphi_0^1}} L_i$. By (28) we have $d_n \in L_{n_{\psi_n \upharpoonright (n-1)}}$ and, by (ii), $n_{\psi_n \upharpoonright (n-1)} \geq n_{\varphi_0^1}$. Thus $d_n \notin \bigcup_{i < n_{\varphi_0^1}} L_i$ and the proof of (27) is finished.

By (vi) and (vii) Λ_n is an antichain in $\langle L, \leq_L \rangle$ and, by (16), (19), (22) and (24), using Fact 2.4 and (11) we conclude that $q_{K_{\varphi \wedge k \wedge j}^{\bigcup_{i < n} \varphi \wedge k} L_i}$, $j \in 2$, are \leq_L -incompatible elements below $q_{K_{\varphi \wedge k}^{\bigcup_{i < n} \varphi} L_i}$. So f_{n+1} is an isomorphism and (vii) is true.

Finally we define d_{n+1} and Φ_{n+1} by

$$d_{n+1} = f_{n+1}(\psi_{n+1}) = f_{n+1}(\varphi_0 \hat{k}_0 \hat{0}) = q_{K_{\varphi_0 \hat{k}_0 \hat{0}}^{\bigcup_{i < n} \varphi_0 \hat{k}_0} L_i} \quad (29)$$

$$\Phi_{n+1} = \bigcup_{\varphi \in {}^{n-1}2 \wedge k \in 2} \left\{ q_{K_{\varphi \wedge k \wedge 0}^{\bigcup_{i < l_{n+1}} L_i}}, q_{K_{\varphi \wedge k \wedge 1 \cup \{d_{n+1}\}}^{\bigcup_{i < l_{n+1}} L_i}} \right\}.$$

Then (ix) is true. By (29) $d_{n+1} \in L_{n_{\varphi_0 \hat{k}_0}}$ and, by (25) $n_{\varphi_0 \hat{k}_0} < l_{n+1}$. Thus Φ_{n+1} is well defined and (x) is true. Claim 2 is proved. \square

Claim 3 Let $\Lambda := \bigcup_{n \in \omega} \Lambda_n$, $f := \bigcup_{n \in \mathbb{N}} f_n$ and $D := \{d_n : n \in \mathbb{N}\}$. Then

- (a) The mapping $f : \langle {}^{<\omega}2, \supset \rangle \rightarrow \langle \Lambda, \leq_L \rangle$ is an isomorphism;
- (b) D is a dense subset of the binary reversed tree $\langle \Lambda, \leq_L \rangle$.

Proof. (a) follows from (vii) and (viii).

(b) By (ix) we have $D = f[\Delta]$ and, by Lemma 3.1, Δ is a dense subset of the binary reversed tree ${}^{<\omega}2$. So, by (a), D is a dense set in the poset $\langle \Lambda, \leq_L \rangle$. \square

Claim 4 For each $n \in \mathbb{N}$, $\psi \in {}^{n-1}2$ and $k, j \in 2$ we have

$$q_{K_{\psi \wedge k \wedge j}^{\bigcup_{i < n} \psi \wedge k} L_i} \sim d_n \Leftrightarrow k = 1. \quad (30)$$

Proof. (\Leftarrow) By (iv) we have $q_{K_{\psi \wedge 1 \wedge j}^{\bigcup_{i < n} \psi \wedge 1} L_i} <_L q_{K_{\psi \wedge 1 \cup \{d_n\}}^{\bigcup_{i < l_n} L_i}}$, which, by (8), implies $d_n \in K_{\psi \wedge 1 \wedge j}$. By (L4), $q_{K_{\psi \wedge 1 \wedge j}^{\bigcup_{i < n} \psi \wedge 1} L_i} \in R_{K_{\psi \wedge 1 \wedge j}^{\bigcup_{i < n} \psi \wedge 1} L_i}$ so we have $q_{K_{\psi \wedge 1 \wedge j}^{\bigcup_{i < n} \psi \wedge 1} L_i} \sim d_n$.

(\Rightarrow) By (iii) we have $q_{K_{\psi \wedge 0 \wedge j}^{\bigcup_{i < n} \psi \wedge 0} L_i} <_L q_{K_{\psi \wedge 0}^{\bigcup_{i < l_n} L_i}}$, which, by (8) and (i), implies

$$K_{\psi \wedge 0 \wedge j} \cap \bigcup_{i < l_n} L_i = K_{\psi \wedge 0} \subset \bigcup_{i < n_\psi} L_i. \quad (31)$$

By (17) and (28) we have

$$d_n \in L_{n_{\psi \upharpoonright (n-1)}} \cap \bigcup_{i < l_n} L_i \quad (32)$$

and, by (ii), $n_\psi \leq n_{\psi \upharpoonright (n-1)} < l_n < n_{\psi \wedge 0}$, which implies $d_n \notin \bigcup_{i < n_\psi} L_i$ and, by (31) and (32), $d_n \in \bigcup_{i < l_n} L_i \setminus K_{\psi \wedge 0 \wedge j} \subset \bigcup_{i < n_{\psi \wedge 0}} L_i \setminus K_{\psi \wedge 0 \wedge j}$. Now, since $q_{K_{\psi \wedge 0 \wedge j}^{\bigcup_{i < n} \psi \wedge 0} L_i} \in R_{K_{\psi \wedge 0 \wedge j}^{\bigcup_{i < n} \psi \wedge 0} L_i}$, we have $q_{K_{\psi \wedge 0 \wedge j}^{\bigcup_{i < n} \psi \wedge 0} L_i} \not\sim d_n$. Claim 4 is proved. \square

Claim 5 For each $\varphi \in {}^{<\omega}2 \setminus \{\emptyset\}$, each $j \in 2$, and each $k \in \text{dom } \varphi$ we have

$$q_{K_{\varphi \frown j}}^{\bigcup_{i < n_\varphi} L_i} \sim d_{k+1} \Leftrightarrow \varphi(k) = 1. \quad (33)$$

Proof. Let $k \in \text{dom } \varphi$. Then there is $s \in 2$ such that $(\varphi \upharpoonright k) \frown \varphi(k) \frown s \subset \varphi \frown j$ and by (vii) we have $f(\varphi \frown j) \leq_L f((\varphi \upharpoonright k) \frown \varphi(k) \frown s)$, that is,

$$q_{K_{\varphi \frown j}}^{\bigcup_{i < n_\varphi} L_i} \leq_L q_{K_{(\varphi \upharpoonright k) \frown \varphi(k) \frown s}}^{\bigcup_{i < n_{(\varphi \upharpoonright k) \frown \varphi(k)}} L_i} \quad (34)$$

and, by Claim 4,

$$q_{K_{(\varphi \upharpoonright k) \frown \varphi(k) \frown s}}^{\bigcup_{i < n_{(\varphi \upharpoonright k) \frown \varphi(k)}} L_i} \sim d_{k+1} \Leftrightarrow \varphi(k) = 1. \quad (35)$$

By (ii) we have $n_{\psi_{k+1} \upharpoonright k} < l_{k+1} < n_{(\varphi \upharpoonright k) \frown \varphi(k)}$ so, by (32), $d_{k+1} \in L_{n_{\psi_{k+1} \upharpoonright k}} \subset \bigcup_{i < n_{(\varphi \upharpoonright k) \frown \varphi(k)}} L_i$ and, by (34) and Fact 2.4, $q_{K_{\varphi \frown j}}^{\bigcup_{i < n_\varphi} L_i} \sim d_{k+1}$ if and only if $q_{K_{(\varphi \upharpoonright k) \frown \varphi(k) \frown s}}^{\bigcup_{i < n_{(\varphi \upharpoonright k) \frown \varphi(k)}} L_i} \sim d_{k+1}$. Now (33) follows from (35) and Claim 5 is proved. \square

Claim 6 $D \in \mathbb{P}(R)$.

Proof. We show that the subgraph $D = \{d_n : n \in \mathbb{N}\}$ of R satisfies condition (5), that is, for $K \subset H \in [\mathbb{N}]^{<\omega}$ we show that $D \cap R_{\{d_n : n \in K\}}^{\{d_n : n \in H\}} \neq \emptyset$. Let $m := \max H$ and let $\varphi \in {}^m 2$ be defined by

$$\varphi(k) = \begin{cases} 1 & \text{if } k+1 \in K, \\ 0 & \text{if } k+1 \in [1, m] \setminus K. \end{cases} \quad (36)$$

Then, by Claim 5, for $l < m$ we have

$$q := q_{K_{\varphi \frown 0}}^{\bigcup_{i < n_\varphi} L_i} \sim d_{l+1} \Leftrightarrow \varphi(l) = 1. \quad (37)$$

Thus for $l \in K$ by (36) we have $\varphi(l-1) = 1$ and, by (37), $q \sim d_l$; similarly, for $l \in H \setminus K$ by (36) we have $\varphi(l-1) = 0$ and, by (37), $q \not\sim d_l$. So, $q \in R_{\{d_n : n \in K\}}^{\{d_n : n \in H\}}$. Since $q \in \Lambda$ and, by Claim 3(b), D is a dense set in the poset $\langle \Lambda, \leq_L \rangle$, there is $d_r \in D$ such that $d_r \leq_L q_{K_{\varphi \frown 0}}^{\bigcup_{i < n_\varphi} L_i}$. By Fact 2.4 for each $u \in \bigcup_{i < n_\varphi} L_i$ we have: $d_r \sim u$ if and only if $q \sim u$; so it remains to be shown that $\{d_n : n \in H\} \subset \bigcup_{i < n_\varphi} L_i$. But this is true since by (ii) we have $l_m < n_\varphi$ and for $n \in H$ we have $n \leq m$ so, by (17), $d_n \in \bigcup_{i < l_n} L_i \subset \bigcup_{i < l_m} L_i \subset \bigcup_{i < n_\varphi} L_i$. Thus $d_r \in D \cap R_{\{d_n : n \in K\}}^{\{d_n : n \in H\}}$ and Claim 6 is proved. \square

Claim 7 For each $\varphi \in {}^{<\omega}2$ and each $j \in 2$ we have

$$\left(-\infty, q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i} \right)_{\langle L, \leq_L \rangle} \cap D \in \mathbb{P}(R). \quad (38)$$

Proof. Let $K \subset H$ be finite subsets of $(-\infty, q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i})_{\langle L, \leq_L \rangle} \cap D$, where $K = \{d_k : k \in F\}$ and $H \setminus K = \{d_k : k \in G\}$. Then F and G are disjoint finite subsets of \mathbb{N} . By (vii), for $\theta \in {}^{<\omega}2$ and $r \in 2$ we have

$$\varphi \wedge j \not\subseteq \theta \wedge r \Leftrightarrow q_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i} <_L q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}. \quad (39)$$

By the assumption and (28), for $k \in F \cup G$ we have $d_k = q_{K_{(\psi_k \upharpoonright (k-1)) \wedge 0}}^{\bigcup_{i < n_{\psi_k \upharpoonright (k-1)}} L_i} <_L q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}$, which by (39) implies $\varphi \wedge j \not\subseteq (\psi_k \upharpoonright (k-1)) \wedge 0$ and, hence, $|\varphi| + 1 < k$. Thus $k-1 \notin \text{dom}(\varphi \wedge j)$, for each $k \in F \cup G$ and we take $\theta \in {}^{<\omega}2$ such that $H \subset \bigcup_{i < n_\theta} L_i$ and

$$\varphi \wedge j \subset \theta \quad \wedge \quad \forall k \in F \ (\theta(k-1) = 1) \quad \wedge \quad \forall k \in G \ (\theta(k-1) = 0). \quad (40)$$

Let $r \in 2$. By Claim 5 and (40), for each $d_k \in H$ we have $q_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i} \sim d_k$ iff $\theta(k-1) = 1$ iff $k \in F$ iff $d_k \in K$; thus

$$q_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i} \in L_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i} \subset R_K^H. \quad (41)$$

By (40) and (39) we have $q_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i} <_L q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}$ and, since D is dense in $\langle \Lambda, \leq_L \rangle$, there is $d_{k_0} \leq_L q_{K_{\theta \wedge r}}^{\bigcup_{i < n_\theta} L_i}$ which, by (41) and Fact 2.4, implies $d_{k_0} \in R_K^H$. Clearly, $d_{k_0} \in (-\infty, q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i})_{\langle L, \leq_L \rangle} \cap D$ and Claim 7 is proved. \square

Claim 8 For each $n \in \mathbb{N}$ the set $\mathcal{A}_n = \{(-\infty, q)_{\langle L, \leq_L \rangle} \cap D : q \in \Lambda_n\}$, that is

$$\mathcal{A}_n = \left\{ (-\infty, q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i})_{\langle L, \leq_L \rangle} \cap D : \varphi \in {}^{n-1}2 \wedge j \in 2 \right\}$$

is an antichain in $\langle \mathbb{P}(R), \subset \rangle$, maximal below D .

Proof. By (vi) and (vii), Λ_n is a maximal antichain in the reversed tree $\langle \Lambda, \leq_L \rangle$ of size 2^n and, hence, the sets $(-\infty, q)_{\langle L, \leq_L \rangle}$, $q \in \Lambda_n$, are disjoint and, by Claim 7, \mathcal{A}_n is an antichain in $\langle \mathbb{P}(R), \subset \rangle$. Since $\Lambda \setminus \bigcup_{q \in \Lambda_n} (-\infty, q)_{\langle L, \leq_L \rangle} \subset \bigcup_{i \leq n} \Lambda_i$ is a finite set, for $E \in \mathbb{P}(D)$ we have $E_1 = E \setminus \bigcup_{i \leq n} \Lambda_i \in \mathbb{P}(D)$ (see Fact 2.1(a)) and, since $E_1 \subset \bigcup \mathcal{A}_n$, by Fact 2.1(b) E_1 is compatible with some element of \mathcal{A}_n . Claim 8 is proved. \square

Claim 9 The mapping $F : \langle \Lambda, \leq_L \rangle \rightarrow \langle {}^{<\omega}\omega, \supset \rangle$ defined by $F(q_\emptyset^\emptyset) = \emptyset$ and

$$F(q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}) = \left\langle m_{K_{\varphi \wedge j} \cap \bigcup_{i < k} L_i}^k : k \in \{0, \dots, n_\varphi\} \right\rangle, \quad (42)$$

for $\varphi \in {}^{<\omega}2$ and $j \in 2$, is an embedding.

Proof. By (i) we have $K_{\varphi \wedge j} \subset \bigcup_{i < n_\varphi} L_i$ thus $K_{\varphi \wedge j} \cap \bigcup_{i < k} L_i \subset \bigcup_{i < k} L_i$, for all $k \leq n_\varphi$, and, by (11), $m_{K_{\varphi \wedge j} \cap \bigcup_{i < k} L_i}^k \in \omega$. Thus F is well defined.

For a proof that F is an injection we suppose that $q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i} \neq q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i}$.

If $n_\varphi \neq n_\psi$ then the corresponding sequences are of different length and, hence, different.

Let $n_\varphi = n_\psi$. Suppose that $|\varphi| \neq |\psi|$, say $|\varphi| < |\psi|$. Then, by (ii), $n_\varphi < l_{|\varphi|+1} \leq l_{|\psi|} < n_\psi$, which contradicts our assumption. Thus $\varphi, \psi \in {}^{n-1}2$, for some $n \in \mathbb{N}$.

If $\varphi = \psi$, then, for example $j = 0$ and $l = 1$ and, by (v), $m_{K_{\varphi \wedge 0}}^{n_\varphi} \neq m_{K_{\varphi \wedge 1}}^{n_\varphi}$ so the corresponding sequences have the last elements different.

If $\varphi \neq \psi$, then there are $k < n - 1$ and $\eta \in {}^{k-1}2$ such that, for example, $\eta \wedge 0 \subset \varphi$ and $\eta \wedge 1 \subset \psi$. Since $\eta \wedge 0 \subset \varphi \wedge j$ and $\eta \wedge 1 \subset \psi \wedge l$ by (vii) we have $q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i} \leq_L q_{K_{\eta \wedge 0}}^{\bigcup_{i < n_\eta} L_i}$ and $q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i} \leq_L q_{K_{\eta \wedge 1}}^{\bigcup_{i < n_\eta} L_i}$, which by (8) implies

$$K_{\varphi \wedge j} \cap \bigcup_{i < n_\eta} L_i = K_{\eta \wedge 0} \quad \text{and} \quad K_{\psi \wedge l} \cap \bigcup_{i < n_\eta} L_i = K_{\eta \wedge 1}. \quad (43)$$

By (v) we have $m_{K_{\eta \wedge 0}}^{n_\eta} \neq m_{K_{\eta \wedge 1}}^{n_\eta}$ so, by (43), $m_{K_{\varphi \wedge j} \cap \bigcup_{i < n_\eta} L_i}^{n_\eta} \neq m_{K_{\psi \wedge l} \cap \bigcup_{i < n_\eta} L_i}^{n_\eta}$, which shows that $F(q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}) \neq F(q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i})$. So, F is an injection indeed.

For a proof that F is order preserving we suppose that $q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i} \leq_L q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i}$. Then $n_\varphi \geq n_\psi$ and $K_{\varphi \wedge j} \cap \bigcup_{i < n_\psi} L_i = K_{\psi \wedge l}$ so, for each $k \leq n_\psi$ we have $K_{\varphi \wedge j} \cap \bigcup_{i < k} L_i = K_{\varphi \wedge j} \cap \bigcup_{i < n_\psi} L_i \cap \bigcup_{i < k} L_i = K_{\psi \wedge l} \cap \bigcup_{i < k} L_i$ which implies

$$\left\langle m_{K_{\psi \wedge l} \cap \bigcup_{i < k} L_i}^k : k \leq n_\psi \right\rangle \subset \left\langle m_{K_{\varphi \wedge j} \cap \bigcup_{i < k} L_i}^k : k \leq n_\varphi \right\rangle \quad (44)$$

that is $F(q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}) \supset F(q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i})$. Thus the mapping F is order preserving.

Suppose that (44) holds and that $q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i} \not\leq_L q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i}$. Then, since F is one-to-one, $q_{K_{\psi \wedge l}}^{\bigcup_{i < n_\psi} L_i} \not\leq_L q_{K_{\varphi \wedge j}}^{\bigcup_{i < n_\varphi} L_i}$ and, by (vii), $\varphi \wedge j$ and $\psi \wedge l$ are incompatible elements of ${}^{<\omega}2$. Hence there is $\eta \in {}^{<\omega}2$ such that, for example, $\eta \wedge 0 \subset$

$\varphi \hat{\cap} j$ and $\eta \hat{\cap} 1 \subset \psi \hat{\cap} l$ and, like in the proof that F is an injection we obtain $m_{K_{\varphi \hat{\cap} j} \cap \bigcup_{i < n_\eta} L_i}^{n_\eta} \neq m_{K_{\psi \hat{\cap} l} \cap \bigcup_{i < n_\eta} L_i}^{n_\eta}$, which contradicts (44) (by (ii), $\eta \subset \psi$ implies $n_\eta \leq n_\psi$). Claim 9 is proved. \square

By Claims 3(a) and 9 the composition $g = F \circ f : \langle {}^{<\omega}2, \supset \rangle \rightarrow \langle {}^{<\omega}\omega, \supset \rangle$, where

$$\langle {}^{<\omega}2, \supset \rangle \cong_f \langle \Lambda, \leq_L \rangle \hookrightarrow_F \langle {}^{<\omega}\omega, \supset \rangle$$

is an embedding and by Lemma 3.2 we have $\mathcal{T} := g[{}^{<\omega}2] \uparrow \in \text{Bt}({}^{<\omega}\omega)$. Clearly,

$$\mathcal{T} = \left\{ \psi \in {}^{<\omega}\omega : \exists \varphi \in {}^{<\omega}2 \exists j \in 2 \ \psi \subset \langle m_{K_{\varphi \hat{\cap} j} \cap \bigcup_{i < k} L_i}^k : k \leq n_\varphi \rangle \right\}.$$

Claim 10 $D \Vdash \tau \restriction \check{n} \in \check{\mathcal{T}}$, for each $n \in \omega$.

Proof. Let $n \in \omega$. We have to prove that

$$D \Vdash \exists \varphi \in {}^{<\omega}2 \exists j \in 2 \left(\check{n} \leq n_\varphi \wedge \forall k < \check{n} \ \tau(k) = m_{K_{\varphi \hat{\cap} j} \cap \bigcup_{i < k} L_i}^k \right). \quad (45)$$

Let G be a generic filter and $D \in G$. By (ii), there is $n_0 \in \mathbb{N}$ such that $n < l_{n_0-1}$ and, by (ii) again,

$$\forall \varphi \in {}^{n_0-1}2 \ n_\varphi > l_{n_0-1} > n. \quad (46)$$

By Claim 8, the set $\mathcal{A}_{n_0} = \{(-\infty, q_{K_{\varphi \hat{\cap} j}}^{\bigcup_{i < n_\varphi} L_i})_{\langle L, \leq_L \rangle} \cap D : \varphi \in {}^{n_0-1}2 \wedge j \in 2\}$ is an antichain in $\langle \mathbb{P}(R), \subset \rangle$, maximal below D and, since $D \in G$, there are $\varphi_0 \in {}^{n_0-1}2$ and $j_0 \in 2$ such that

$$\left(-\infty, q_{K_{\varphi_0 \hat{\cap} j_0}}^{\bigcup_{i < n_{\varphi_0}} L_i} \right)_{\langle L, \leq_L \rangle} \cap D \in G. \quad (47)$$

By (46), for $k < n$ we have $k < n_{\varphi_0}$ and, hence, $q_{K_{\varphi_0 \hat{\cap} j_0}}^{\bigcup_{i < n_{\varphi_0}} L_i} \leq_L q_{K_{\varphi_0 \hat{\cap} j_0}}^{\bigcup_{i < k} L_i}$.

So, by Fact 2.4, $(-\infty, q_{K_{\varphi_0 \hat{\cap} j_0}}^{\bigcup_{i < n_{\varphi_0}} L_i})_{\langle L, \leq_L \rangle} \cap D \subset (-\infty, q_{K_{\varphi_0 \hat{\cap} j_0}}^{\bigcup_{i < k} L_i})_{\langle L, \leq_L \rangle}$
 $= L_{K_{\varphi_0 \hat{\cap} j_0} \cap \bigcup_{i < k} L_i}^{\bigcup_{i < k} L_i}$ and, by (11), $L_{K_{\varphi_0 \hat{\cap} j_0} \cap \bigcup_{i < k} L_i}^{\bigcup_{i < k} L_i} \Vdash \tau(\check{k}) = (m_{K_{\varphi_0 \hat{\cap} j_0} \cap \bigcup_{i < k} L_i}^k)^\sim$,
 which, together with (47), gives $\tau_G(k) = m_{K_{\varphi_0 \hat{\cap} j_0} \cap \bigcup_{i < k} L_i}^k$. (45) is proved. \square

Now, since $D \subset \Lambda \subset L \subset B$, (10) follows from Claims 6 and 10. \square

4 Appendix

In this section we establish the mentioned forcing-free translation of the 2-localization property to the language of Boolean algebras. We recall that a complete Boolean algebra \mathbb{B} is said to have the 2-localization property iff

$$1_{\mathbb{B}} \Vdash_{\mathbb{B}} \forall x : \check{\omega} \rightarrow \check{\omega} \ \exists \mathcal{T} \in ((\text{Bt}(<^{\omega}\omega))^V)^{\vee} \ \forall n \in \check{\omega} \ x \restriction n \in \mathcal{T}. \quad (48)$$

Theorem 4.1 *A complete Boolean algebra \mathbb{B} has the 2-localization property if and only if for each double sequence $[b_{nm} : \langle n, m \rangle \in \omega \times \omega]$ of elements of \mathbb{B} we have*

$$\bigwedge_{n \in \omega} \bigvee_{m \in \omega} b_{nm} = \bigvee_{\mathcal{T} \in \text{Bt}(<^{\omega}\omega)} \bigwedge_{n \in \omega} \bigvee_{\varphi \in \mathcal{T} \cap^{n+1}\omega} \bigwedge_{k \leq n} b_{k\varphi(k)}. \quad (49)$$

Proof. For notational simplicity, let b and c denote the left hand side and the right hand side of (49) respectively.

First we prove that $b \geq c$ holds for each sequence $[b_{nm}]$ in any complete Boolean algebra (and, hence, that the equality (49) is equivalent to the inequality $b \leq c$). For $\mathcal{T} \in \text{Bt}(<^{\omega}\omega)$ and $n \in \omega$ we have: $\bigwedge_{k \leq n} b_{k\varphi(k)} \leq b_{n\varphi(n)}$, for each $\varphi \in \mathcal{T} \cap^{n+1}\omega$. Thus $\bigvee_{\varphi \in \mathcal{T} \cap^{n+1}\omega} \bigwedge_{k \leq n} b_{k\varphi(k)} \leq \bigvee_{\varphi \in \mathcal{T} \cap^{n+1}\omega} b_{n\varphi(n)} \leq \bigvee_{m \in \omega} b_{nm}$. This holds for all $n \in \omega$ and, hence,

$$\bigwedge_{n \in \omega} \bigvee_{\varphi \in \mathcal{T} \cap^{n+1}\omega} \bigwedge_{k \leq n} b_{k\varphi(k)} \leq \bigwedge_{n \in \omega} \bigvee_{m \in \omega} b_{nm} = b.$$

This holds for all $\mathcal{T} \in \text{Bt}(<^{\omega}\omega)$ so we have $c \leq b$. Now we prove the theorem.

(\Leftarrow) Assuming that (49) holds for each double sequence $[b_{nm}]$ in \mathbb{B} we prove that \mathbb{B} has the 2-localization property. Let G be a \mathbb{B} -generic filter over V and $x \in V_{\mathbb{B}}[G]$, where $x : \omega \rightarrow \omega$. Let τ be a \mathbb{B} -name and $b_0 \in G$, where $x = \tau_G$ and $b_0 \Vdash \tau : \check{\omega} \rightarrow \check{\omega}$. Defining $b_{nm} := \|\tau(\check{n}) = \check{m}\| = \|\langle \check{n}, \check{m} \rangle \in \tau\|$ we obtain $b_0 \leq \|\forall n \in \check{\omega} \ \exists m \in \check{\omega} \ \langle n, m \rangle \in \tau\| = \bigwedge_{n \in \omega} \bigvee_{m \in \omega} b_{nm} = c$, which means that $b_0 \Vdash \exists \mathcal{T} \in ((\text{Bt}(<^{\omega}\omega))^V)^{\vee} \ \forall n \in \check{\omega} \ \exists \varphi \in \mathcal{T} \cap^{n+1}\omega \ \forall k \leq n \ \tau(k) = \varphi(k)$. So, since $b_0 \in G$, there is $\mathcal{T} \in \text{Bt}(<^{\omega}\omega) \cap V$ such that for each $n \in \omega$ we have $\tau \restriction (n+1) \in \mathcal{T}$. Clearly, $\tau \restriction 0 \in \mathcal{T}$ and we are done.

(\Rightarrow) Suppose that the algebra \mathbb{B} has the 2-localization property. For a double sequence $[b_{nm}]$ in \mathbb{B} we prove (49), that is, $b \leq c$. If $b = 0$, we are done. Otherwise, defining the sequence $[b_{nm}^*]$ by $b_{n0}^* = b_{n0}$ and $b_{nm}^* = b_{nm} \setminus \bigvee_{i < m} b_{ni}$, for $m > 0$, for each $n \in \omega$ we have

$$b \leq \bigvee_{m \in \omega} b_{nm}^* \\ m_1 \neq m_2 \Rightarrow b_{nm_1}^* \wedge b_{nm_2}^* = 0.$$

Hence for a \mathbb{B} -name

$$\tau := \left\{ \left\langle \langle n, m \rangle^\sim, b_{nm}^* \right\rangle : \langle n, m \rangle \in \omega \times \omega \right\}$$

we have $b \Vdash \tau : \check{\omega} \rightarrow \check{\omega}$ and $\|\tau(\check{n}) = \check{m}\| = b_{nm}^*$, which, by the assumption, implies that $b \Vdash_{\mathbb{P}} \exists \mathcal{T} \in ((\text{Bt}(<^\omega \omega))^V)^\sim \forall n \in \check{\omega} \ x \upharpoonright (n+1) \in \mathcal{T}$. Thus

$$b \leq \bigvee_{\mathcal{T} \in \text{Bt}(<^\omega \omega)} \bigwedge_{n \in \omega} \bigvee_{\varphi \in \mathcal{T} \cap^{n+1} \omega} \bigwedge_{k \leq n} b_{k\varphi(k)}^* \leq c,$$

because $b_{nm}^* \leq b_{nm}$, for all $n, m \in \omega$, and the proof is over. \square

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